Proceedings of Inverse Problems Symposium
East Lansing, Michigan, USA
11-12 June 2007

MULTIDIMENSIONAL INVERSE HEAT CONDUCTION CALCULATIONS

Hans-Jürgen Reinhardt*
Department of Mathematics
University of Siegen
57068 Siegen, GERMANY
Email: hjreinhardt@math.uni-siegen.de

Jörg Frohne
Department of Mathematics
University of Siegen
57068 Siegen, GERMANY

INTRODUCTION

Inverse Heat Conduction Problems (IHCPs) have been extensively studied over the last 50 years. They have numerous applications in many branches of science and technology. The problem consists in determining the temperature and heat flux at inaccessible parts of the boundary of a 2- or 3-dimensional body from corresponding data – called 'Cauchy data' – from accessible parts of the boundary. It is well-known that IHCPs are illposed which means that small perturbations in the data may cause large errors in the solution. The importance of inverse heat conduction problems and appropriate solution algorithms are established in numerous works (see, e.g. [1], [3], [5], [6] and the references therein).

In this contribution we give an overview over our contributions to multidimensional IHCP's and indicate what computational results for which sort of problems we have obtained. In [4] we have established the theoretical background for multidimensional inverse heat conduction problems. The solutions of the associated direct problem as well as the inverse problem are understood in weak sense (see [4]). This allows the Cauchy data to be only from L_2. The initial condition can be given or not. In the latter case, our method is also able to identify the initial temperature distribution.

In a 2d-setting our problem is indicated in Figure 1.

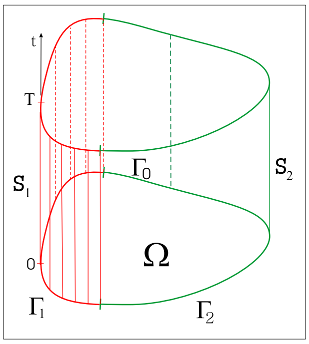

Figure 1. PROBLEM SETTING

REGULARIZATION

The idea of our method is very simple: since the initial condition v_0 and the heat flux $v_1 = \partial u/\partial N|_{S_2}$ at the inaccessible part Γ_2 of the boundary are not known, we consider them as a control $v = (v_0, v_1)$ to minimize the defect $J_0(v) = 1/2\|u(x,\cdot)|_{S_1} - \varphi(\cdot)\|^2_{L_2(S_1)}$ on the accessible part Γ_1 of the boundary. On Γ_0 and Γ_1 different kinds of boundary conditions may be imposed, e.g. homogeneous Neumann boundary conditions on Γ_0 and Dirichlet boundary conditions at Γ_1. The function φ represents the (measured) temperature data on Γ_1. Here, we set $S_i = \Gamma_i \times (0,T], i = 0,1,2$, where $T > 0$ is the final time.

In [4] we have proved the existence of the optimal control, and also obtained the gradient of the defect functional by means of an appropriate adjoint problem. Since the optimal control problem is still unstable, we have to use a regularization method for it.

We solve the underlying inverse problem by discretization in combination with Tikhonov's regularization using a zeroth order penalty term as well as iterative regularization via an appropriate stopping rule. As the underlying operator we choose the Neumann-to-Dirichlet mapping $A : L_2(\Omega) \times L_2(S_2) \longrightarrow L_2(S_1)$ which maps the (unknown) initial function v_0 and the heat flux $v_1 = \partial u/\partial N|S_2$ to $u|_{S_1}$ where u is the solution of the heat equation in weak form. The minimizing functional to determine $v = (v_0, v_1)$,

*Address all correspondence to this author.

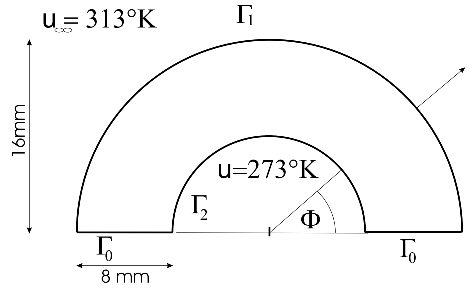

Figure 2. IHCP ON A HALF RING

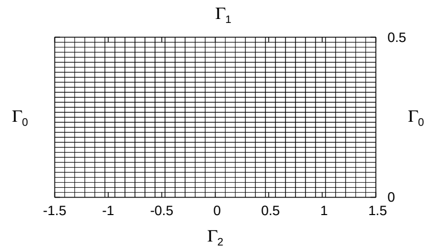

Figure 3. RECTANGLE WITH 33 GRID POINTS ON Γ_1 and Γ_2

$$J_\gamma(v) = \frac{1}{2}\left(\|Av-\varphi\|^2_{L_2(S_1)} + \gamma^2\|v\|^2_{L_2(\Omega)\times L_2(S_2)}\right) \qquad (1)$$

is differentiable with gradient $J'_\gamma(v) = A^*(Av-\varphi)+\gamma^2 v$ which can be obtained via the solution of an appropriate adjoint problem.

As an iterative algorithm to solve the minimization problem we use the Conjugate Gradient Method (CGM) in connection with an appropriate stopping rule. This is also called CGLS (=conjugate gradient least square) in [8]. We allow perturbations of the data, $\|\varphi-\varphi_\varepsilon\|_{L_2(S_1)} = O(\varepsilon)$, and the operator A is replaced by A_h which is a finite element or finite difference approximation.

NUMERICAL EXAMPLES

Our program is based on a C++ code of C. Fröbel [7] where, as a direct solver for the underlying parabolic problems, the Finite Element package DEAL [2] is used. For the inverse problem calculations we use the *Crank-Nicolson method* for the time integration and the conjugate gradient method plus Tikhonov's regularization to solve the minimization problem described above. The direct solution of the parabolic problems with DEAL uses bilinear ansatz functions. Our computational results are discussed in detail in [9].

As examples we consider an IHCP on a half ring (see Fig. 2) and on a rectangle (see Fig. 3).

REFERENCES

[1] O.M. Alifanov, Inverse Heat Transfer Problems, Springer, 1994.

[2] W. Bangerth, R. Hartmann, G. Kanschat, deal.II Differential Equations and Analysis Library. http://www.dealii.org, 2006

[3] J. V. Beck, B. Blackwell and C. R. St-Clair Jr., Inverse Heat Conduction: Ill-Posed Problems, Wiley Intersciences, New York, 1985.

[4] Dinh Nho Hào, H.-J. Reinhardt, Y. Jarny, A variational method for multi-dimensional linear inverse heat conduction problems. Matimyas matematika (Special Issue) (1998), 48-56.

[5] Dinh Nho Hào, Methods for Inverse Heat Conduction Problems. Peter Lang Verlag, Frankfurt/Main, Bern, New York, Paris, 1998.

[6] Dinh Nho Hào and H.-J. Reinhardt, Gradient methods for inverse heat conduction problems. Inverse Problems in Engineering **6**, No. 3 (1998), 177-211.

[7] C. Fröbel, Tikhonov-Regularisierung zur Parameteridentifizierung bei elliptischen und parabolischen Modellgleichungen. Diplom Thesis, Zentrum für Technomathematik, Univ. Bremen, 2004.

[8] A. Frommer, P. Maass, Fast cg-based methods for Tikhonov-Phillips regularization. SIAM J. Sci. Comput. **20**, No. 5 (1999), 1831 - 1850.

[9] H.-J. Reinhardt, J. Frohne, F.-T. Suttmeier, Dinh Nho Hào, Numerical solution of Inverse Heat Conduction Problems in two spatial dimensions. J. Inverse and Ill-Posed Problems **15** (2007), 19-36.